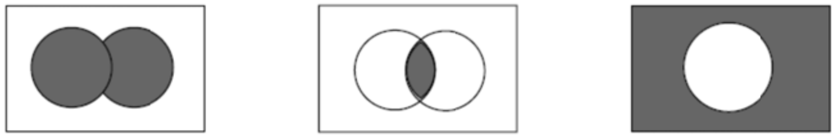
Operations in partially ordered sets, and some linguistic applications

Anna Szabolcsi, March 7, 2013 (short guest talk in L. Champollion's seminar)

(1) union: $A \cup B$ intersection: $A \cap B$ complement: $-A$
disjunction: $p \vee q$ conjunction: $p \wedge q$ negation: $\neg p$

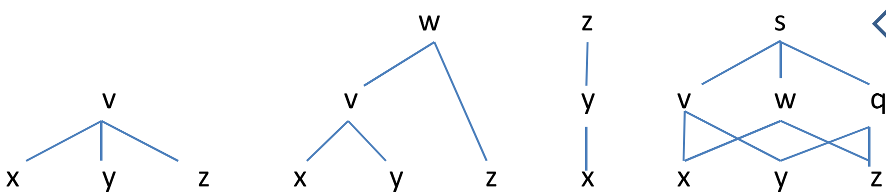


What other operations are these related to? On what kind of entities can such operations be performed? What kind of structures do these entities form?

- Partially ordered set (poset):
  < A, ≥ > where ≥ is reflexive, transitive, and anti-symmetrical

Join, ∨ the least upper bound of a two-element set: a∨b
Meet, ∧ the greatest lower bound of a two-element set: a∧b

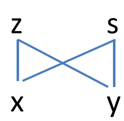
- Join semi-lattices: posets that are closed under join (viz., for any a,b ∈ A, a∨b ∈ A):



⟸ This last poset is a free join semi-lattice: it has all imaginable distinct joins

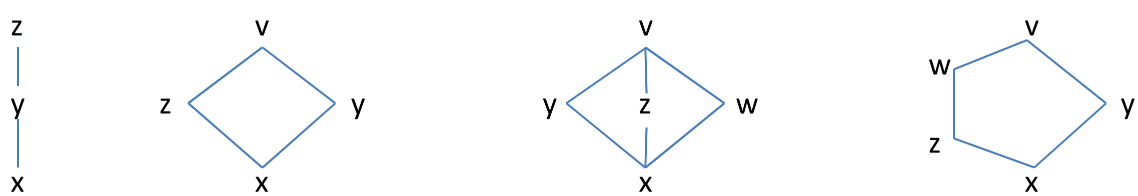
Mereological structures are free join semi-lattices

A poset that is not closed under join
(e.g. there is no x∨y in A)
not a join semi-lattice:



- Similarly for meet semi-lattices (same, upside down).

- A poset that is closed under both meet and join is a lattice.
  A lattice that has a top, T and a bottom, ⊥ is bounded. (All finite lattices are bounded.)



- If the meet operation preserves non-empty finite joins, and the join-operation preserves non-empty finite meets, the lattice is distributive. The first two lattices above are distributive, the diamond and the pentagon are not.
  a∨(b∧c) = (a∨b) ∧ (a∨c) and a ∧ (b∨c) = (a∧b) ∨ (a∧c)

- Complement, $\neg$ for any $a \in A$, $\neg a$ is another element of A for which both of these hold: $a \wedge \neg a = \bot$ and $a \vee \neg a = T$

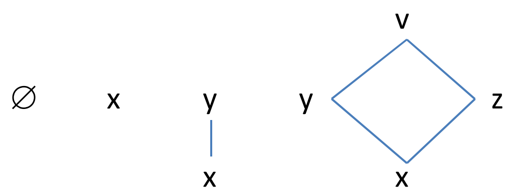
- Relative pseudo-complement, $\rightarrow$ $c \in A$ is $a \rightarrow b$, the pseudo-complement of a relative to b, iff c is the largest element of A for which $(a \wedge c) \leq b$.

- If a poset is closed under meet, join, and unique complement, it is a Boolean algebra.
- If a poset is closed under meet, join, and rel. pseudo-complement, it is a Heyting algebra.

Every Boolean algebra and every Heyting algebra is a distributive lattice.
Every Boolean algebra is a Heyting algebra with $a \rightarrow \bot = \neg a$.

A Heyting algebra that is not a Boolean algebra:

Some (Heyting algebras that are also) Boolean algebras:



among others, every finite powerset algebra

- Some logico-linguistic applications:

Classical propositional logic is a Boolean algebra (connectives: meet, join, and complement).
Logics with no excluded middle, hence no double negation cancellation, are modeled with Heyting algebras. Intuitionistic logic, the logics of Dynamic Semantics and Inquisitive Semantics.
Inquisitive Semantics: each element of A is a Hamblinian question (a set of sets of worlds); informative content vs. inquisitive content.
Events and collectives form mere join semi-lattices (no $\bot$, no lattice).

An algebraic semantics of scope-taking (Szabolcsi & Zwarts 1993):
If the meaning of a scopal element is (at least in part) defined in terms of Boolean operations, cash out its contributions by performing those operations on the denotation of its scope.

What didn't you see?	compl. of {x: you saw x})
What did every girl see?	meet of { {x: Mary saw x}, {x: Susan saw x}, ... }
What did a(ny) girl see?	join of { {x: Mary saw x}, {x: Susan saw x}, ... }
What did two girls see?	[distributive two requires both meets and joins]

Predicts trouble when the scope of the operator denotes an element of some A that is not closed under the pertinent operations: Weak Island effects. E.g. collectives and events (and hence <event,object> pairs) form join semi-lattices: not closed under meet or complement.

Which relatives of yours did you show every one of your rings to?	OK wh > every
Which relatives of yours did you get every one of your rings from?	# wh > every

4,000 people visited the Rijksmuseum last year	OK 4,000 events, altogether 40 persons
4,000 people didn't visit the Rijksmuseum last year	# 4,000 events, altogether 40 persons